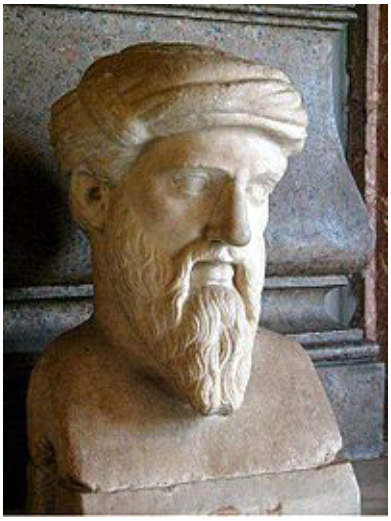
Abel Prize 2010

Number Theory; the mathematical play-ground of John Tate

Algebraic integers, finite fields, p-adic numbers. Class field theory, rigid analytic spaces, elliptic curves. These are some of the concepts you must know if your task is to describe the mathematical achievements of John Torrence Tate, the Abel Laureate 2010.
If you don' t know anything about any of the listed words, you can still share something with one of the most brilliant scientist of our time; the fascination for the natural numbers. At the very first glance, they look innocent and easily accessible. Counting, 1, 2, 3, ..., or computing, 1+2=3, 3+5=8, it's a childs game. But as you learn more about them, you realize that the world you are diving into is huge, mysteriuos and unpredictable.

"Mathematics is the queen of the sciences and number theory is the queen of mathematics."

Carl Friedrich Gauss



"Number is the within of all things."

Pythagoras of Samos

"God invented the integers; all else is the work of man."

Leopold Kronecker

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Number theory is the study of the properties of numbers in general, and integers in particular. Number theory may be subdivided into several fields, according to the methods used and the type of questions investigated.

Algebraic number theory

In algebraic number theory, the concept of a number is expanded to algebraic numbers, i.e. roots of polynomials with rational coefficients. These domains contain elements analogous to the integers, the so-called algebraic integers, which is a main subject of study in this field.

Many number theoretic questions are attacked by reduction modulo p for various primes p. This localization procedure leads to the construction of the p-adic numbers, another main subject of study in the field of algebraic number theory.

Elementary number theory

In elementary number theory, integers are studied without use of techniques from other mathematical fields. Important discoveries of this field are Fermat's little theorem, Euler's theorem, the Chinese remainder theorem and the law of quadratic reciprocity, to mention a few.

Analytic number theory

Analytic number theory employs the machinery of calculus and complex analysis to tackle questions about integers. Examples include the prime number theorem concerning the asymptotic behavior of the primes and the Riemann hypothesis, but also proofs of the transcendence of π or e, are classified as analytical number theory.

Arithmetic algebraic geometry

Arithmetic (algebraic) geometry is the study of schemes of finite type over the spectrum of the ring of integers **Z**.

Diophantine geometry

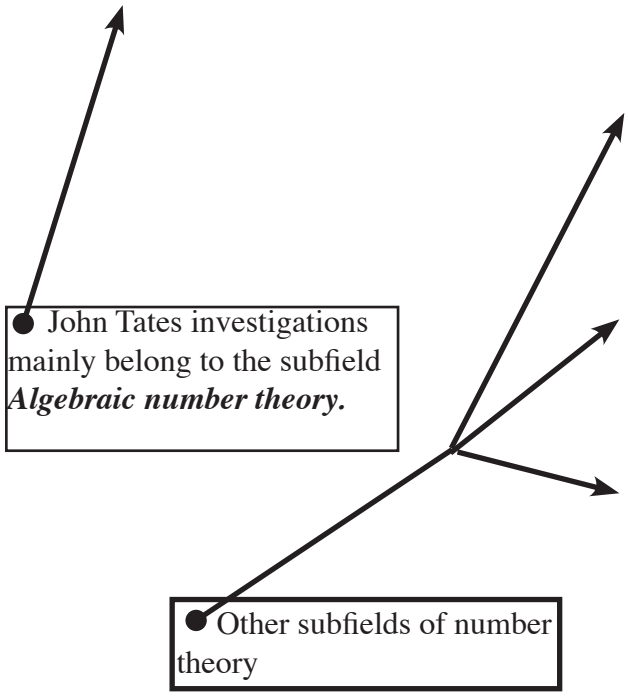
Diophantine geometry is the study of algebraic varieties over number fields.

Combinatorial number theory

Combinatorial number theory deals with number theoretic problems which involve combinatorial ideas in their formulations or solutions. Paul Erdös is the main founder of this branch of number theory. Examples are the problems of finding arithmetic progressions in a set of integers.

Modular forms

Modular forms are analytic functions on the upper half-plane satisfying a certain kind of functional equation and a growth condition. The theory of modular forms therefore belongs to complex analysis but the main importance of the theory has traditionally been in its connections with number theory.



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John Tate´s influence in modern number theory can be read out of the numerous results and concepts named after him. Here are some of them:

Hodge-Tate theory; p-adic analogue of the Hodge decomposition for complex cohomology.

Tate module; a Galois module constructed from an abelian variety over a field

The ***Lubin–Tate formal group law*** is the unique (1-dimensional) formal group law F such that $e(x) = px + x^p$ is an endomorphism of F, i.e. such that $e(F(x,y)) = F(e(x), e(y))$

The ***Tate Isogeny theorem*** says that abelian varieties with isomorphic Tate modules are isogenous.

The ***Sato–Tate conjecture*** is a statistical statement about the family of elliptic curves E_p over the finite field with p elements, with p a prime number, obtained from an elliptic curve E over the rational number field, by the process of reduction modulo a prime for almost all p.

Tate twist; a particular abelian group with an action of a Galois group constructed from a field.

In the theory of elliptic curves, ***Tate's algorithm,*** takes as input an integral model of an elliptic curve E over **Q** and a prime p. It returns the exponent f_p of p in the conductor of E, the type of reduction at p, and the local index c_p.

Tate motive is the tensor inverse of the Lefschetz motive

The ***Tate-Shafarevich group***, named for Tate and Igor Shafarevich, of an abelian variety defined over a number field K consists of the elements of the Weil-Châtelet group that become trivial in all of the completions of K

Via the ***Serre-Tate theorem*** one can control (part of) the char p deformations of an abelian scheme coming from the local part of the Barsotti-Tate group.

Barsotti-Tate groups; arise in "nature" when one consider the sequence of kernels of multiplication by successive powers of p on an abelian variety.

Néron–Tate height (or canonical height) is a quadratic form on the Mordell-Weil group of rational points of an abelian variety defined over a global field.

Tate cohomology groups are a slightly modified form of the usual cohomology groups of a finite group that combine homology and cohomology groups into one sequence.

Honda-Tate theory; i.e. classification of abelian varieties over finite fields up to isogeny.

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A fundamental result in number theory is the Unique-Prime-Factorization Theorem for integers. In 1847, in an attempt to prove Fermats Last Theorem, Gabriel Lamé incorrectly assumed that this property holds in general. He was immediately corrected by Joseph Liouville who referred to results of Ernst Kummer about failure of unique prime factorization in certain rings of algebraic integers, published in 1843.
This innocent little dispute became the origin of a branch of number theory, in which John Tate has been a main figure during the last 50 years.

Prime factorization in algebraic number fields

In number theory the starting points is the set of integers, ..., -3, -2, -1, 0, 1, 2, 3, ..., denoted by **Z**. The integers are included in the rational numbers **Q**, i.e. all fractions of integers, where the denominator is different from 0. Unfortunately the number $\sqrt{2}$, defined as the root of the polynomial equation $x^2-2=0$ is not included in **Q**, as observed by the Pythagoreans around 400 BC. Nevertheless, we are interested in studying the properties of $\sqrt{2}$, so we extend **Q** by $\sqrt{2}$ to obtain our first homemade algebraic number field, denoted $\mathbf{Q}(\sqrt{2})$, consisting of all number which can be written as $a+b\sqrt{2}$, for rational numbers a and b.
The number $\sqrt{2}$ is defined as the solution of certain polynomial equation. One can in fact show that all numbers in $\mathbf{Q}(\sqrt{2})$ satisfy some polynomial equation. Not the same one for all, but at least one for each.
Some rational numbers are integers, and some of the algebraic numbers are algebraic integers. The way we decide if a number should be called an integer, is as follows; we look at the monic polynomial equation for the number (monic means that the coefficient of the highest degree term is 1). If all the coefficients are integers, all roots of the equation are algebraic integers, If not, they are not! Examples of algebraic integers in $\mathbf{Q}(\sqrt{2})$ are $\sqrt{2}$ (root of the polynomial x^2-2) and $1+\sqrt{2}$ (root of x^2-2x-1).

The Fundamental Theorem of Arithmetic

The Fundamental Theorem of Arithmetic (or Unique-Prime-Factorization Theorem) states that any integer greater than 1 can be written as a unique product (up to ordering of the factors) of prime numbers. Intuitively, this theorem characterizes prime numbers uniquely in the sense that they are the ”fundamental numbers.” The theorem was practically proved by Euclid but the first full and correct proof is found in the Disquisitiones Arithmeticae by Carl Friedrich Gauss.

A fundamental property of the integers is the unique factorization property. There is only on way of writing 105 as a product of primes ($105=3\cdot5\cdot7$) when we do not bother about the order of the factors. But in a general algebraic number field this is no longer true. The favourite example for (nearly) all mathematicians is the extension of **Q** by the square root of -5. (If you have bad feelings for the square root of a negative number, just close your eyes and keep walking. You will get used to it.) In this extension, or rather the integral part of it, the number 6 has two different prime factorizations.

$$6=2\cdot3=(1+\sqrt{-5})\cdot(1-\sqrt{-5})$$

All factors involved are prime numbers, i.e. only divisible by 1 and itself.
The extent to which unique factorization fails in the ring of integers of an algebraic number field can be described by a certain group known as a class group. If the class group is finite, then the order of the group is called the class number. The class group of the algebraic integers of an algebraic number field is trivial if and only if the ring of algebraic integers has the unique factorization property. The size of the ideal class group can thus be considered as a measure for the deviation from being a unique factorization domain.